Synchronization in the presence of memory

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Abstract

We study the effect of memory on synchronization of identical chaotic systems driven by common external noises. Our examples show that while in general synchronization transition becomes more difficult to meet when memory range increases, for intermediate ranges the synchronization tendency of systems can be enhanced. Generally the synchronization transition is found to depend on the memory range and the ratio of noise strength to memory amplitude, which indicates on a possibility of optimizing synchronization by memory. We also point out on a close link between dynamics with memory and noise, and recently discovered synchronizing properties of networks with delayed interactions.

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Since a generic study of Fahy and Hamman [1] synchronization of dynamical systems has become an active field of research. Examples of synchronous behavior are found in physical, biological, chemical and social systems [2], from a road traffic anticipation [3], population growth [4] and secure communications [5] through biophysics [6], chemistry [7] and laser optics [8] to computer science [9]. At least four types of synchronization scenarios have been identified [10] of which the synchronization between identical systems coupled by a common noise has received much attention due to its relative simplicity and importance [11].

In recent years studies of synchronization were extended further to obey an important case of interactions that are delayed in time [12, 13, 14]. One of the most striking observations here was enhancement of synchrony for networks of many nonlinear interacting units by time-delayed transmission of the signals [14, 15, 16, 17]. More specifically, in a neural network model [15] an enhanced synchronization of neurons by delays has been observed. Similar effect was found for a network of coupled logistic maps [16]. Even a network of logistic maps with random delay times was able to sustain synchronization [17].

A purpose of this work is to show that the constructive influence of delays on synchronization can be extended further to obey a broad class of nonlinear systems with noise and memory, where the latter is understood as the auto-feedback with delay having distribution in time. A close link between the synchronizing networks under delay and synchronization by a common noise of (effectively decoupled) nonlinear systems with memory is also demonstrated. Detailed numerical calculations are carried out for generalized logistic maps and chaotic Fahy-Hamman systems described by non-Markovian Langevin equations. In both cases the choice of the models has been motivated by their well documented synchronizing properties in the limit of vanishing memory. Importantly, our analysis demonstrates that the presence of memory and noise not only can sustain synchronization existing in equivalent, memoryless systems, but also can enhance it. The models chosen, though governed by different dynamics, collectively display this possibility, which suggests that the effect can be quite common for nonlinear interacting systems.

Memory and randomness naturally link with time evolution of interacting dynamical systems [18, 19, 20]. Indeed, the evolution of a system coupled with 'external' degrees of freedom (*e.g.* open systems, nonlinear networks) can, at least in principle, be reduced to a dynamics of an effective single system, but with memory and noise. The effective system is usually more amenable to numerical, analytical and formal considerations. Perhaps the

simplest and exact example of such reduction is that of a nonlinear system coupled bilinearly to harmonic oscillators. The elimination of the oscillator degrees of freedom results in a generalized Langevin equation for the dynamics of the system, where memory and noise terms are fully specified by the properties of the oscillators and by their coupling with the system [18]. Similarly, the dynamics of a network with delay times can be approximated by, or in some cases reduced to an effective dynamics of single nodes with memory and noise. Consider, for example, a network of coupled logistic maps with discrete time [16, 17]. The state $x_i(t+1)$ of the node "i" at time $t+1$ depends on the state $x_i(t)$ of that node at time t and on the states $x_j(t-\tau_{ij})$ of the nodes {"$j \neq i$"} that couple to "i" at earlier times {t-τ_{ij}}, where $\{\tau_{ij}\}$ are the delay times (see *e.g* Eq. (1) in [16, 17]). By iterating the equations for $x_j(t-\tau_{ij})$ $(j \neq i)$ back in time and re-substituting them to $x_i(t+1)$ we arrive at the effective, single-node equations for $x_i(t+1)$ expressed in terms of $x_i(t)$ and the nonlinear auto-feedback (memory) Γ. For the network of logistic maps Γ is a polynomial in $x_i(t-n)$, $1 \leq n \leq t$ with coefficients depending on network's connectivity and the initial values $x_j(0)$ for the nodes. Any randomness in the original network like random delay times, random elements in network's connectivity, or averaging over $x_j(0)$ goes into additive/multiplicative noise terms in the effective equations of motion.

One of the simplest, but important, class of Γs is a linear auto-feedback with $\Gamma \sim \sum_{k=1}^{N} \Gamma_k x_i(t-k)$, which for example, can represent a 'mean-dynamics' of the above mentioned networks of logistic maps. For Γ given by a polynomial in $x_i(t-n)$ a recipe for getting $\{\Gamma_k\}$ of the 'mean-dynamics' would be *e.g.* a replacement of $x_i(t-n)$ by $x_i(t-n) =< x > +[x_i(t-n)- < x >] =< x > +\delta x_i(t-n)$ and neglect of terms that are nonlinear in δx_i; $\langle\cdot\rangle$ denotes the average over the trajectory and over initial conditions.

The discussion as given clearly shows that memory and noise are intrinsic to dynamical evolution of a system. It is then important to know in what way they affect synchronization. We report on the noise-induced synchronization, which is generic for this case. The synchronizing system is characterized by master trajectories that divide the whole phase space onto basins of attraction such that all trajectories initiated in the same basin and subjected to the same sequence of the noise evolve to the same master trajectory [1, 9].

As our first model we consider an ensemble of chaotic logistic maps coupled by common, additive noises. Since the coupling is realized only through the noise terms it is sufficient to

explore just two such systems, which we define as:

$$x_{n+1}^i = 4x_n^i(1 - x_n^i) + I\sum_{k=1}^{N}\Gamma_k x_{n-k}^i + \xi_n + \epsilon_n^i \mod 1, \tag{1}$$

where $i = 1, 2$ and ξ_n is the non-symmetric, δ-correlated noise taken uniformly from the interval $[a, a + b]$. To avoid roundoff-induced synchronization we also add an extremely small independent uniform noise ϵ_n^i (from interval $[-10^{-12}, 10^{-12}]$) to each particle at every iterated step [21]. We extend our analysis to a symmetric version of (1) by defining a new variable

$$z_n^i = x_n^i - \langle x \rangle \,, \tag{2}$$

where $\langle x \rangle \equiv \langle x_n^i \rangle$. For $I = 0$ the noise-induced synchronization in the $[a, b]$-plane has recently been studied in detail by Rim *et al.* [22]. We restrict ourselves to linear auto-feedback with $\Gamma = I\sum_{k=1}^{N}\Gamma_k x_{n-k}$. The set of coefficients $\{I\Gamma_k\}$ is the 'memory profile' and N is proportional to the memory range. Two models for the memory profile are considered in detail: the constant memory profile with

$$\Gamma_n \equiv \Gamma_n^c = \begin{cases} 1 & \text{for} \quad n \leq N, \\ 0 & \text{for} \quad n > N, \end{cases} \tag{3}$$

and the exponentially decaying memory profile

$$\Gamma_n \equiv \Gamma_n^e = \begin{cases} \exp(-\lambda n) & \text{for} \quad n \leq N, \\ 0 & \text{for} \quad n > N. \end{cases} \tag{4}$$

For $N \gg 1/\lambda$ the inversion of λ is the memory range.

The stability of the synchronized states is determined by the sign of the (maximal) transversal Lyapunov exponent Λ for the dynamics of difference $\delta x_n = x_n^1 - x_n^2 = z_n^1 - z_n^2$. In the numerical calculations we choose at random the initial states $\{x_N^1, x_{N-1}^1, ..., x_0^1\}$ from the allowed interval and nearby states $\{x_N^2, x_{N-1}^2, ..., x_0^2\}$. Then we iterate the equations of motion to construct statistics of the expansion and contraction rates: $\lambda_i = \ln\left(\frac{|\delta x_{N+i+1}|}{|\delta x_{N+i}|}\right)$ of δx_n. The procedure, repeated for many randomly chosen initial states, allows us to calculate the average of λ_i, which approximates Λ [13, 15, 22].

In case of non-zero memory we generally find that for large enough absolute noise intensity, $|I|$, the synchronization is destroyed for all $N \geq 1$. Results are shown in Fig. 1 where $|I|$, above which the synchronization region disappears, is sketched. For $|I|$ exceeding the

threshold value the systems de-synchronize for all a and b. Choosing $\lambda = \frac{2}{N}$ we find that the results compare well for both memory profiles. In this case Γ_N^e at $n = N$ is about an order of magnitude smaller than at $n = 0$ ($\Gamma_N^e = e^{-2}$). The effect is illustrated further in Fig. 2,

Figure 1: Threshold lines for symmetric map (2): (a) Γ_n^e, $I < 0$, (b) Γ_n^e, $I > 0$, (c) Γ_n^c, $I < 0$, and (d) Γ_n^c, $I > 0$. For Γ_n^e cases we take $\lambda = \frac{2}{N}$.

where the evolution of the synchronizing boundaries, $\Lambda(a, b) = 0$, are shown with increasing (positive) intensity for the symmetric map (2) and for different memory profiles. The case without memory [22] is also shown for comparison. Please note that the synchronization area shrinks with increasing intensity and range of the memory. This behavior is observed for $N \geq 1$, for positive and negative intensities, and for all maps studied. Interestingly, the

Figure 2: Synchronization areas for symmetric map. Region (a) corresponds to system without memory [22]. Left plot is done for Γ_n^c with (b) $N = 5$, $I = 0.1$ and (c) $N = 5$, $I = 0.3$. Right plot corresponds to Γ_n^e with $\lambda = \frac{2}{N}$ and with (d) $N = 5$, $I = 0.1$ and (e) $N = 5$, $I = 0.5$.

maxima in Fig. 1 prove that *memory can also act on synchronization in a constructive way by enhancing it.* We observe the enhancement of synchrony by memory for both memory

profiles, which indicates that the phenomenon can be quite general. Indeed, as demonstrated below systems with a more complex dynamics, governed by the integrodifferential, generalized Langevin equation, show similar behavior. We discuss the possible implications of these results toward the end.

Memory profile is closely linked to a time-time autocorrelation function

$$C_k = \frac{\langle (x_n - \langle x \rangle)\,(x_{n-k} - \langle x \rangle) \rangle}{\left\langle (x_n - \langle x \rangle)^2 \right\rangle}. \tag{5}$$

We monitored the behavior of this function inside- and outside of the synchronizing area. Exemplary results are shown in Figs. 3 and 4. The decay time τ in Fig. 4 was calculated by assuming that C_k is a linear combination of trigonometric functions multiplied by an exponential decay of the form $|C_k| = \exp\left(-\frac{k}{\tau}\right)$. We observe in Figs. 2 and 4, that close to the border of the synchronizing area τ is enhanced by the presence of memory, with the maximum positioned outside this area. That is, the chaotic systems with larger τ 'remember for longer' about their initial conditions, which makes synchronization more difficult and explains intuitively the shrinkage of the areas in Fig. 2.

Figure 3: Correlation functions for the symmetric map with constant memory (${C^c}_k$) and without memory (${C^0}_k$). Here $N = 5$, $I = 0.1$, $b = 0.1$, $a = 5.3$ ($a = 5.55$ for insets).

The second model with which we explore influence of memory on noise-induced synchronization is the generalization of the Fahy-Hamman system [1]. We analyze trajectories of two identical particles in a two dimensional potential well given by:

$$V(x_1, x_2) = \frac{\sin 2\pi x_1}{2\pi x_1} + \frac{\sin 2\pi x_2}{2\pi x_2} + \frac{(x_1^2 + x_2^2)^2}{16\pi^2}, \tag{6}$$

Figure 4: Decay time of C_k as a function of noise parameter a. Cases studied are: (a) constant memory and (b) exponential memory for $I = 0.1$ and $\lambda = 2/N$. The case (c) corresponds to I=0. The remaining parameters are $N = 5$ and $b = 0.1$.

with different initial conditions. The motion of the i-th particle ($i = 1, 2$) is governed by the generalized Langevin equation:

$$m\ddot{x}^i_\alpha(t) = -\frac{\partial V(x^i_1, x^i_2)}{\partial x^i_\alpha} - mI \int^t \Gamma(t-t')\; \dot{x}^i_\alpha(t')\; dt' + \xi(t), \tag{7}$$

where m is the mass, I is again the memory intensity or, in this case, the friction constant. The noise ξ, common for both particles, is a Γ-correlated stochastic force with zero mean, where correlations are obeying fluctuation-dissipation theorem:

$$\langle\langle\, \xi(t)\xi(t')\, \rangle\rangle = 2mIk_BT\; \Gamma(t-t'), \tag{8}$$

with T being the absolute temperature and k_B the Boltzmann constant. In what follows we restrict ourselves to the exponentially correlated noise by choosing, as in Eq. (4), $\Gamma(t-t') = e^{-\lambda(t-t')}$. Double angular brackets denote averaging over a noise realization. The equations of motion (7) are integrated numerically using stochastic version of the Euler algorithm. Discretization of the equations (7) entails re-scaling the noise strength by a factor $1/\sqrt{\Delta t}$, where Δt is the time step. Finally, the exponentially correlated noise is generated from uniformly distributed random numbers through Ornstein-Uhlenbeck process. As previously, simulations are carried out to determine the maximal Lyapunov exponent, Λ, as function of memory range ($1/\lambda$) and I. We used a natural system of units: energy $\epsilon_u = V(0,0) - V_{min} \approx 2.41$, time $t_u = \sqrt{3}$ measuring curvature of the potential at origin and length $l_u = 1$ giving the period of the oscillating part of the potential. In the absence of memory the

system was originally studied by Fahy and Hamman [1] using regular Andersen thermostat. The results unambiguously showed that trajectories were exponentially convergent to a common trajectory after a transient period. The same phenomenon has been reported for a Langevin dynamics without memory of a one-dimensional Lennard-Jones chain [23] and of other systems [9].

Calculations of Λ in the presence of memory as function of $1/\lambda$ are shown in Fig. 5. Please note that the dependence of Λ on $1/\lambda$ is much more complex now than that previously observed for the maps. At least three regimes can be identified. In the first regime, corresponding to a short memory range, we observe de-synchronization of the system by memory. However, after reaching maximum, Λ drops down and for intermediate memory range synchronization is considerably enhanced. The strongest synchronization conditions are met for $1/\lambda \approx 0.14$, where Λ approaches minimum. Interestingly, the positions of the extremes are practically independent on I.

Figure 5: Maximal Lyapunov exponent Λ against memory strength $1/\lambda$ for three different friction constants I and the reduced temperature $k_B T = 1$.

Summarizing, the results obtained for the generalized logistic maps and for the dynamical system evolving according to the generalized Langevin equation uncover a possibility of having the *constructive* influence of memory on the noise-induced synchronization. The results are quite counterintuitive for, in the first place, we would expect that memory by introducing extra dimensions [24] should act in just the opposite way, *i.e.* making synchronization more difficult [16, 17]. Though the proposed models are relatively simple they represent quite different dynamics, which suggests that the uncovered enhancement of synchrony by

memory can be a general phenomenon, occurring for a wide class of nonlinear dynamical systems with memory and noise. They rise a possibility of seeking for a right memory profile to create optimal conditions for synchronization to occur *i.e. optimizing synchronization by memory.* Additionally, our findings apart from giving yet another example of the nontrivial interplay between memory and noise, show a close relation to the synchronizing properties of coupled networks with delays that have recently been discovered.

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